Econometrics I
TA Session 1

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1 Course Aims and Objectives

This class is TA session of Econometrics I. Our course aim is to review the contents of the main class, but mainly explain the exercises solution and related R exercises.

2 Matrix Knowledge

Paying attention to the vectors' or matrices' dimension is vital for their calculation.

2.1 Some General Rules for Matrix Calculation

Consider the case of matricies $A \in \mathbb{R}^{2\times 2}$ and $B \in \mathbb{R}^{2\times 2}$:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; \tag{1}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}. \tag{2}$$

In this case, we can define the calculation of them as follows:

$$A \pm B = \begin{pmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} \end{pmatrix}; \tag{3}$$

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}. \tag{4}$$

Let A' denotes the transpose matrix of A:

$$A' = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}. \tag{5}$$

In the main lectures, Prof. Poignard may let $A^\top$ denotes the transpose matrix, but I prefer A'.

The general rules for matrix multiplication are explained as follows.

- Associative Law: $(XY)Z = X(YZ)$
- Distributive Law: $X(Y + Z) = XY + XZ$
- $(XY)' = Y'X'$
- $(XYZ)' = Z'Y'X'$

Also, the idempotent matrix is defined as follows.

Definition 2.1. If the matrix $X \in \mathbb{R}^{n \times n}$ satisfies $X^2 = XX = X$, X is an **idempotent matrix**. When X is a symmetric idempotent matrix, then $X'X = X$.

2.2 Rank

The relationship of the rank and matrix is represented as follows:

Definition 2.2. The **row(column) rank** of matrix $X \in \mathbb{R}^{n \times k}$ is the dimension of the vector space that is spanned by its row(column) vectors.

In other words, we can check the $rank(X)$ by counting the maximum numbers of linearly independent row(column) vectors. For example, consider the following matrix $D \in \mathbb{R}^{3 \times 4}$:

$$D = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 3 \\ 0 & 0 & 3 & 3 \end{pmatrix}. \tag{6}$$

Let us denote each row vector as $d_1 = (1\ 2\ 0\ 1)$, $d_2 = (2\ 4\ 1\ 3)$ and $d_3 = (0\ 0\ 3\ 3)$ respectively.

$$c_1 d_1 + c_2 d_2 = 0 \Leftrightarrow c_1 = c_2 = 0. \tag{7}$$

If above equation hold, then d_1 and d_2 are linearly independent. However, d_1, d_2 and d_3 are linearly dependent because we obtain $d_3 = 3d_2 - 6d_1$. Therefore, we have $rank(D) = 2$. Additionaly, let us denote each column vector as $e_1 = (1\ 2\ 0), e_2 = (2\ 4\ 0), e_3 = (0\ 1\ 3)$ and $e_4 = (1\ 3\ 3)$ respectively.

$$c_3 e_1 + c_4 e_3 = 0 \Leftrightarrow c_3 = c_4 = 0. \tag{8}$$

However, e_1, e_3 and e_4 are lineary dependent. Generally, the raw rank and the column rank are same value.

2.3 Trace

The trace of a square matrix $X \in \mathbb{R}^{n \times n}$ is derived by the sum of its diagonal elements:

$$tr(X) = \sum_{j=1}^{k} x_{jj}. \tag{9}$$

Following results are shown.

(a) $tr(cX) = c(tr(X))$

(b) $tr(X') = tr(X)$

(c) $tr(X+Y) = tr(X) + tr(Y)$

(d) $tr(XY) = tr(YX)$

(e) $x'x = tr(x'x) = tr(xx')$ if x is a vector

2.4 Sums of Values

Denote a vector whose all elements are 1 as $i \in \mathbb{R}^{n\times 1}$. Then, sum of the elements of a vector $y \in \mathbb{R}^{n\times 1}$ is represented as:

$$\sum_{j=1}^{n} y_j = \underset{1\times n \times n\times 1}{i'y}. \tag{10}$$

From this operation, we can rewrite the arithmetic mean of y as:

$$\bar{y} = \frac{1}{n}\sum_{j=1}^{n} y_j = \frac{1}{n} i'y. \tag{11}$$

Consider the matrix $X \in \mathbb{R}^{n\times k}$:

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n1} & \cdots & x_{nk} \end{pmatrix}. \tag{12}$$

Then, we can represnt a design matrix $X'X$ by using its row vector $x_i = (x_{i1}, x_{i2}, \ldots, x_{ik})$ and its transpose matrix x_i' [*1]:

$$X'X = \sum_{i=1}^{n} x_i' x_i. \tag{13}$$

These operations are necessary calculation when you estimate a regression model. I will explain some conditions for the invertibility of this matrix later.

2.5 Inverse Matrix

If the square matrix X is **the regular matrix**, there is an inverse matrix of X.

[*1] Greene(2011) denotes $X'X$ by using its column vector formed by the transpose of the low vector. Therefore, the row vector is represented as x_i' and we can rewrite $X'X$ as $X'X = \sum_{i=1}^{n} x_i x_i'$. However, this confusing notation is not used in the main textbook, so we premise (13) unless otherwise noted.

Definition 2.3. Suppose the case of matrix $X \in \mathbb{R}^{n\times n}$. If there is a matrix X^{-1} which introduces $XX^{-1} = I$, X^{-1} is called as **an inverse matrix** of X.

Suppose the case of $A \in \mathbb{R}^{2\times 2}$ in (1). The inverse matrix of A is calculated as follows:

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}. \tag{14}$$

When you want to find the inverse matrix $X^{-1} \in \mathbb{R}^{n\times n}(n \geq 3)$, the row reduction method(行基本変形) and the cofactor expansion(余因子展開) are usually used. These topics are explained in some of the mathematical textbooks such as Chiang and Wainwright(2006) "Fundamental Methods of Mathematical Economics(4th edition)", McGraw-Hill.

Finallly, following theorem is important.

Theorem 2.1. Following three conditions are equivalent:

- X is a regular matrix.
- The determinant of X is non-zero.
- The row vectors of X are linearly independent.

2.6 Patitioned Matrix

Sometimes, partitioning the matrix to make some groups of elements is useful. For instance, we can make blocks of the elements of matrix $X \in \mathbb{R}^{4\times 4}$ as follows:

$$X = \left(\begin{array}{cc|cc} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ \hline x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{array}\right) = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{21} \end{pmatrix}$$

Suppose that the matrix $Y \in \mathbb{R}^{4\times 4}$ is patitioned like X. For these matricies X and Y,

$$X \pm Y = \begin{pmatrix} X_{11} \pm Y_{11} & X_{12} \pm Y_{12} \\ X_{21} \pm Y_{21} & X_{22} \pm Y_{22} \end{pmatrix}, \tag{15}$$

and

$$XY = \begin{pmatrix} X_{11}Y_{11} + X_{12}Y_{21} & X_{11}Y_{12} + X_{12}Y_{22} \\ X_{21}Y_{11} + X_{22}Y_{21} & X_{21}Y_{12} + X_{22}Y_{22} \end{pmatrix}. \tag{16}$$

are satisfied.

2.7 Kronecker Products

Consider the matricies $X = (x_{ij}) \in \mathbb{R}^{m\times n}$ and $Y = (y_{ij}) \in \mathbb{R}^{p\times q}$ respectively. Then, the **Kronecker product** is defined as:

$$X \otimes Y = (x_{ij}Y) \in \mathbb{R}^{mp\times nq}. \tag{17}$$

The extended form of (17) becomes:

$$X \otimes Y = \begin{pmatrix} x_{11}Y & x_{12}Y & \cdots & x_{1n}Y \\ x_{21}Y & x_{22}Y & \cdots & x_{2n}Y \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1}Y & x_{m2}Y & \cdots & x_{mn}Y \end{pmatrix} \in \mathbb{R}^{mp\times nq}. \tag{18}$$

Thus, the (i,j)th partition of this matrix is represented, with the partitioned matrix, as $x_{ij}Y$. This definition of Kronecker product gives rise to the following results.

Theorem 2.2. The following results hold:

(a) $X_1X_2 \otimes Y_1Y_2 = (X_1 \otimes Y_1)(X_2 \otimes Y_2)$,
(b) $(X \otimes Y)^{-1} = X^{-1} \otimes Y^{-1}$ if the inverse exists,
(c) $(X \otimes Y)' = X' \otimes Y'$,
(d) $(X \otimes Y)(X^{-1} \otimes Y^{-1}) = I$.

2.8 Eigenvalue, Eigenvector, and Diagonalisation

2.8.1 Eigenvalue and Eigenvectors

Let a matrix $A \in \mathbb{R}^{k\times k}$ be square. If a scalar λ and a vector $c \in \mathbb{R}^{k\times 1}$, which is normalised as $c'c = 1$, satisfy the following equation, then they are called as the **eigenvalue (characteristic root** and the **eigenvector (characteristic vector)** respectively.

$$Ac = \lambda c \tag{19}$$

Rewrite above equation as:

$$Ac - \lambda c = 0; \tag{20}$$
$$(A - \lambda I)c = 0. \tag{21}$$

A necessary and sufficient condition to derive the non-zero solution with respect to λ is the characteristic equation of A:

$$det|A - \lambda I| = 0. \tag{22}$$

A solution of (22) are k eigenvalues including equal roots. each $(\lambda_1, \lambda_2, ..., \lambda_k)$ corresponds to eigenvalues $(c_1, c_2, \ldots, c_k)$. Here, we consider a simple example. Suppose a matrix A as:

$$A = \begin{pmatrix} 8 & 1 \\ 4 & 5 \end{pmatrix}. \tag{23}$$

Then, we can calculate characteristic roots as follows.

$$\begin{aligned} det|A - \lambda I| &= (8-\lambda)(5-\lambda) - 4 \\ &= 36 - 13\lambda + \lambda^2 \\ &= (\lambda - 9)(\lambda - 4) = 0 \end{aligned} \tag{24}$$

. By solving above equation, we can derive $\lambda_1 = 9,\ \lambda_2 = 4$ (interchangable). We can obtain λ_1 by following calculation:

$$\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0. \tag{25}$$

Thus, we can get $c_{11} = c_{12}$ as follows:

$$c = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}. \tag{26}$$

2.8.2 Diagonalisation

A matrix $A \in \mathbb{R}^{k \times k}$ has k distinct characteristic vectors $(c_1, c_2, \ldots, c_k)$ and corresponding characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then, we can make a regular matrix $C = (c_1, c_2, \ldots, c_k)$. This matrix is the orthogonal matrix(直交行列:$C^{-1} = C'$) which diagonalises A.

Definition 2.4. diagonalis **The diagonalisation** of matrix A is defined as follows:

$$C'AC = \Lambda = I\Lambda = C'C\Lambda \tag{27}$$

Λ is called as a diagonalised matrix whose diagonal elements consist of $\lambda_1, \lambda_2, \ldots, \lambda_k$:

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{pmatrix} \in \mathbb{R}^{k \times k} \tag{28}$$

Next, we introduce an useful property of diagonalisation. Firstly, let's check the following results:

$$\begin{aligned} AA' &= (C\Lambda C')(C\Lambda C') \\ &= C\Lambda C' C\Lambda C' \\ &= C\Lambda\Lambda C' \\ &= C\Lambda^2 C'. \end{aligned} \tag{29}$$

We can repeat to premultiply A or its transpose by AA' of (29). This procedure is enable us to apply to the case of non-natural numbers. (We are going to describe in more detail of this procedure in the next subsection.)

2.9 Quadratic Form

Consider a symmetric matrix $A \in \mathbb{R}^{k\times k}$ and a vector $x \in \mathbb{R}^{k\times 1}$. Then, the quadratic form is written as follows:

$$q = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j a_{ij} = x'Ax \tag{30}$$

Generally, q may be positive, negative, or zero. There are some matricies, however, for which q will be positive regardress of x, and others for which q will always be negative. For a given symmetric matrix A,

Definition 2.5.

1. If $x'Ax > 0$ for all nonzero x, then A is a **positive definite matrix**.
2. If $x'Ax < 0$ for all nonzero x, then A is a **negative definite matrix**.
3. If $x'Ax \geq 0$ for all nonzero x, then A is a **positive semidefinite matrix**.
4. If $x'Ax \leq 0$ for all nonzero x, then A is a **negative semidefinite matrix**.

This definition is useful for the optimisation methods. Although it might seem difficult to check a matrix for definiteness, we can easily know it by using the spectral decomposition of A, $A = C\Lambda C'$. Let $y = C'x$. We can rewrite the quadratic form as follows:

$$x'Ax = x'C\Lambda C'x$$
$$= y'\Lambda y$$
$$= \sum_{i=1}^{n} \lambda_i y_i^2. \tag{31}$$

If λ_i is positive for all i, then regardless of y and x, q will be positive. From this calculation, we obtain the following theorem.

Theorem 2.3.

1. Let A be a symmetric matrix. If all the eigenvalues of A are positive (negative), then A is positive definite(negative definite).
2. If some of eigenvalues are zero, then A is positive(negative) semidefinite if the reminder are positive(negative).
3. If A has both negative and positive eigenvalues, then A is indefinite.

Finally, the positive definiteness of a matrix gives rise to the following theorem.

Theorem 2.4. For **a positive definite matrix** A, whose eigenvalues are strictly positive, $A^r = C\Lambda^r C'$, for any real number, r.

The above theorem implies that we can easily find B such that $B^r = A$ by using the diagonalisation. These methods and concepts are important to introduce the GLS estimator.

3 Differentiation of Matrix

Consider the two vectors such as $a \in \mathbb{R}^{n\times 1}$ and $\beta \in \mathbb{R}^{n\times 1}$:

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad and \quad \beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}. \tag{32}$$

Then, we can say $a'\beta = a_1\beta_1 + a_2\beta_2 + \cdots + a_n\beta_n$ and the differentiation of the scalar $a'\beta$ can be described as follows:

$$\frac{\partial a'\beta}{\partial \beta} = \begin{pmatrix} \frac{\partial(a_1\beta_1 + a_2\beta_2 + \cdots + a_n\beta_n)}{\partial \beta_1} \\ \vdots \\ \frac{\partial(a_1\beta_1 + a_2\beta_2 + \cdots + a_n\beta_n)}{\partial \beta_n} \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = a. \tag{33}$$

Definition 3.1. The differentiation of the scalar $a'\beta$ is given by:

$$\frac{\partial a'\beta}{\partial \beta} = a. \tag{34}$$

Next, let $A \in \mathbb{R}^{n\times n}$ and $\beta \in \mathbb{R}^{n\times 1}$.

Definition 3.2. The differentiation of the quadratic form $\beta' A\beta$ is:

$$\frac{\partial \beta' A\beta}{\partial \beta} = (A + A')\beta. \tag{35}$$

We only consider the case that β is 2×1 vector and A is 2×2 matrix.

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \ and\ \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}. \tag{36}$$

In this case, the quadratic form is:

$$\beta' A\beta = a\beta_1^2 + b\beta_1\beta_2 + c\beta_1\beta_2 + d\beta_1^2. \tag{37}$$

Then, derive the differentiation of $\beta' A\beta$ as follows:

$$\begin{aligned} \frac{\partial \beta' A\beta}{\partial \beta} &= \begin{pmatrix} \frac{\partial(a\beta_1^2+b\beta_1\beta_2+c\beta_1\beta_2+d\beta_1^2)}{\partial\beta_1} \\ \frac{\partial(a\beta_1^2+b\beta_1\beta_2+c\beta_1\beta_2+d\beta_1^2)}{\partial\beta_2} \end{pmatrix} \\ &= \begin{pmatrix} 2a\beta_1 + b\beta_2 + c\beta_2 \\ b\beta_1 + c\beta_1 + 2d\beta_2 \end{pmatrix} \\ &= \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \\ &= [\begin{pmatrix} a & c \\ b & d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}] \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \\ &= (A + A')\beta. \end{aligned} \tag{38}$$

Especially, when A is symmetic, we can derive the following result:

$$\frac{\partial \beta' A\beta}{\partial \beta} = (2A)\beta. \tag{39}$$