Exercise

Derive the maximum likelihood estimator of following distributions:

1 Normal distribution

Assuming $X_1, \dots, X_n \overset{i.i.d.}{\sim} N_{\mathbb{R}}(\mu, \sigma^2)$. Derive the MLE of μ and σ^2.

HINT:

$$f_X(x_i) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right)$$

2 Exponential distribution

Assuming $X_1, \dots, X_n \overset{i.i.d.}{\sim}$ Exponential(λ):

$$f_X(x_i) = \lambda \exp(-\lambda x), \quad x \geq 0$$

3 Gumbel distribution

Assuming $X_1, \dots, X_n \overset{i.i.d.}{\sim}$ Gumbel(μ, β):

$$f_X(x_i) = \frac{1}{\beta} \exp\left(-\frac{x - \mu}{\beta}\right) \exp\left(-\exp\left(-\frac{x - \mu}{\beta}\right)\right)$$

4 Poisson distribution

Assuming $X_1, \dots, X_n \overset{i.i.d.}{\sim}$ Poisson(λ):

$$P(X = x_i) = \frac{\lambda_i^x \exp(-\lambda)}{x_i!}, \quad k = 0, 1, \dots, n$$

5 Bernoulli distribution

Assuming $X_1, \dots, X_n \overset{i.i.d.}{\sim}$ Bernoulli(p):

$$P(X = x_i) = p_i^x (1 - p)^{1 - x_i}, \quad x = 0, 1$$

6 Binomial distribution

Assuming $X_1, \dots, X_n \overset{i.i.d.}{\sim}$ Binomial(n, p):

$$P(X = x_i) = \binom{n}{x_i} p_i^x (1 - p)^{n - x_i}, \quad k = 0, 1, \dots, n$$

7 Uniform distribution

Assuming $X_1, \dots, X_n \overset{i.i.d.}{\sim} \text{Uniform}(a, b)$:

$$f_X(x_i) = \frac{1}{b-a}, \quad a \leq x \leq b$$

8 Simple Linear Model

Assuming the following model:

$$y_i = x_i'\beta + u_i,$$

where y_i and u_i are scalars, $x_i \in \mathbb{R}^k$, $\beta \in \mathbb{R}^k$, and for all $i = (1, \dots, n)$, u_i is an i.i.d. random variable following a normal distribution with mean μ and variance σ^2. Derive the MLEs.

9 Multivariate Version

Assuming the following model:

$$y = X\beta + u,$$

where $y \in \mathbb{R}^n$, $u \in \mathbb{R}^n$, $X \in \mathbb{R}^{n \times k}$, $\beta \in \mathbb{R}^k$, and u follows a multivariate normal distribution with mean 0 and variance $\sigma^2 I \in \mathbb{R}^{n \times n}$. Derive the MLEs.