Econometrics I
TA Session 15

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1 Large Sample Tests

If you are interested in this topic, you had better read Newey and McFadden(1994).

1.1 Framework

We consider an M–estimator $\hat{\theta}$ satisfying below criterion:

$$\hat{\theta} = \arg\max_{\theta} \ \mathbb{L}_n(\theta),$$

1.2 Intuitive Representation

Let $\theta = (\theta_1, \theta_2)'$ and the null hypothesis as follows:

$$H_0 \colon \theta_2 = \theta_{c,2},$$

where the index "c" represents constraint.
For example, consider an unconstrained estimator:

$$\hat{\theta} = \begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{pmatrix} = \arg\max_{\theta} \ \mathbb{L}_n(\theta),$$

and a constrained one:

$$\hat{\theta}^0 = \begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2^{constr} \end{pmatrix} = \arg\max_{\theta} \ \mathbb{L}_n(\theta_1, \theta_2^{constr}).$$

1.3 Three Tests

There are mainly three tests for large number tests. They are a little bit different from each other, but they provide the same results. Researchers mainly use the Wald test because of its user-friendliness.

- Likelihood ratio test: If the restricted model is adequate, then the difference between the maximized objection functions $l(\hat{\theta}) - l(\hat{\theta}^0)$ should not significantly differ from zero.
- Score test (Lagrange multiplier test): If the restricted model is adequate, then the slope of the tangent of the log–likelihood function at the restricted MLE should not significantly differ from zero, which is the slope of the tangent of the log–likelihood function at the unrestricted MLE.
- Wald test: If the restricted model is adequate, then the restriction function evaluated at the unrestricted MLE should not significantly differ from zero, which is the value of the restriction function at the restricted MLE.

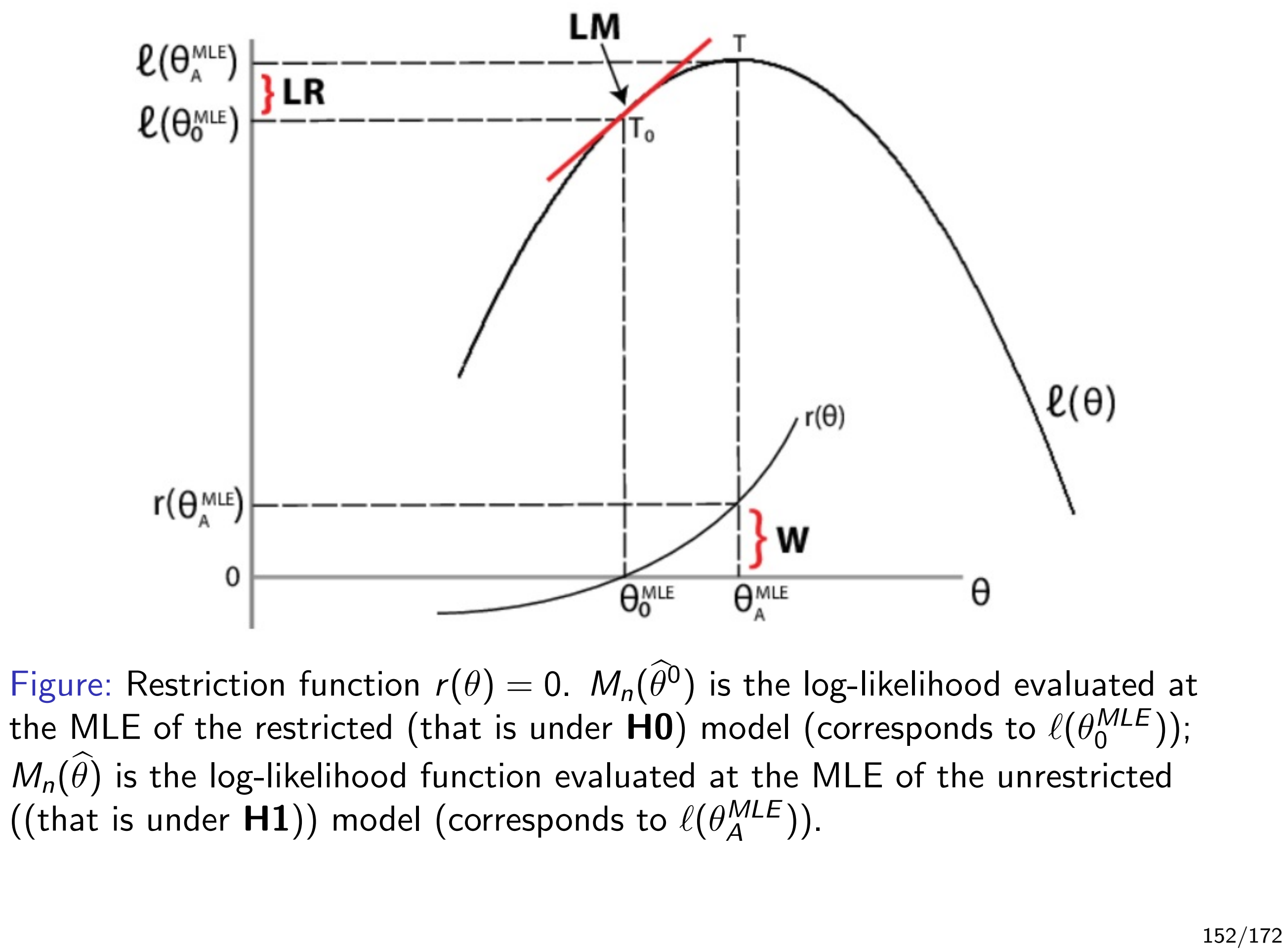

Figure 1: *
This is from the slide of "Advanced Econometrics" made by B. Poignard.

2 The Wald Test

The **null hypothesis** is assumed to be:

$$\mathbf{H0}\colon r(\theta_0) = \mathbf{0}$$

where $r\colon \mathbb{R}^K \to \mathbb{R}^M$ with $M \leq K$. Intuitively, we want to check if $r(\hat{\theta})$ is close to zero. Let

$$\hat{\theta} = \arg\max_{\theta} \ \mathbb{L}_n(\theta) = \arg\max_{\theta} \ \frac{1}{n}\Sigma_{i=1}^{n} \log f_\theta(y_i, x_i).$$

Then, we have the following theorem.

Theorem 2.1. Under **H0**, the Wald statistic

$$\zeta_n^W = nr'(\hat{\theta})\Sigma_W^{-1}(\hat{\theta})r(\hat{\theta})$$

is asymptotically distributed as $\chi^2(M)$, with

$$\Sigma_W(\theta) = \nabla_{\theta'} r(\hat{\theta})J^{-1}(\theta)I(\theta)J^{-1}(\theta)(\nabla_\theta r(\theta))'$$
$$\Sigma_W(\hat{\theta}) = \nabla_{\theta'} r(\hat{\theta})\hat{J}^{-1}(\hat{\theta})\hat{I}(\hat{\theta})\hat{J}^{-1}(\hat{\theta})(\nabla_\theta r(\hat{\theta}))'$$

and $J(\theta) = \mathbb{E}[\nabla^2_{\theta\theta'} \log f_\theta(y_i, x_i)]$ and $I(\theta) = \mathbb{V}[\nabla_\theta \log f_\theta(x_i, y_i)]$ with $\hat{J}(\hat{\theta})$ and $\hat{I}(\hat{\theta})$ consistent estiamtes of $J(\theta)$ and $I(\theta)$. The test with critical region $W_n = \{\zeta_n^W \geq \chi^2_{1-\alpha}(M)\}$ has an asymptotic level $1 - \alpha$.

3 The Score Test (Lagrange Multiplier Test)

We denote by $\hat{\theta}^0$ the constrained estimator. Intuitively, we want to check if $\nabla_\theta \mathbb{L}_n(\hat{\theta}^0)$ is close to zero. We assume here $I(\theta) = J(\theta)$. Then, the following theorem holds.

Theorem 3.1. Under **H0**, the score statistic

$$\zeta_n^S = n\nabla_{\theta'}\mathbb{L}_n(\hat{\theta}^0)\hat{J}^{-1}(\hat{\theta}^0)\nabla_\theta\mathbb{L}_n(\hat{\theta}^0)$$

satisfies

$$\zeta_n^S = \zeta_n^W + o_p(1).$$

In particular, under **H0**, the asymptotic distribution of ζ_n^S is $\chi^2(M)$ and the test with critical region $S_n = \{\zeta_n^S \geq \chi^2_{1-\alpha}(M)\}$ has an asymptotic level $1 - \alpha$.

4 The Likelihood Ratio Test

Intuitively, we want to check if $\mathbb{L}_n(\hat{\theta}) - \mathbb{L}_n(\hat{\theta}^0)$ is close to zero. We assume here $I(\theta) = J(\theta)$. Then, the following theorem holds.

Theorem 4.1. Under **H0**, the statistic

$$\zeta_n^R = 2n\Big(\mathbb{L}_n(\hat{\theta}) - \mathbb{L}_n(\hat{\theta}^0)\Big)$$

satisfies

$$\zeta_n^R = \zeta_n^W + o_p(1) = \zeta_n^S + o_p(1).$$

In particular, under **H0**, the asymptotic distribution of ζ_n^R is $\chi^2(M)$ and the test with critical region $LR_n = \{\zeta_n^R \geq \chi^2_{1-\alpha}(M)\}$ has an asymptotic level $1 - \alpha$.

5 Summary of the Three Tests

Summary.

- Likelihood ratio test: Value of the maximized unrestricted log–likelihood $l(\hat{\theta})$; value of the maximized restricted log–likelilhood $l(\hat{\theta}^0)$; number of restrictions (degrees of freedom: M); Test statistic:

$$\zeta_n^R = 2n\Big(l(\hat{\theta}) - l(\hat{\theta}^0)\Big).$$

- Score test (Lagrange multiplier test): Gradient of log–likelihood evaluated at the restricted MLE $\nabla_\theta l(\hat{\theta}^0)$; variance–covariance matrix evaluated at the restricted MLE; Test statistic:

$$\zeta_n^S = n\nabla_{\theta'}\mathbb{L}_n(\hat{\theta}^0)\hat{J}^{-1}(\hat{\theta}^0)\nabla_\theta \mathbb{L}_n(\hat{\theta}^0)$$

- Wald test: Restriction function evaluated at the unrestricted MLE $\hat{\theta}$; The Jacobian of the restriction function evaluated at the unrestricted MLE $\hat{\theta}$; variance–covariance matrix evaluated at the unrestricted MLE; Test statistic:

$$\zeta_n^W = nr'(\hat{\theta})\Sigma_W^{-1}(\hat{\theta})r(\hat{\theta}),$$

with

$$\Sigma_W(\hat{\theta}) = \nabla_{\theta'}r(\hat{\theta})\hat{J}^{-1}(\hat{\theta})\hat{I}(\hat{\theta})\hat{J}^{-1}(\hat{\theta})(\nabla_\theta r(\hat{\theta}))'.$$

The three tests are asymptotically equivalent. Under the null, the LR, LM, and Wald test statistics are all distributed as $\chi^2(M)$, with M the degree of freedom equal to the number of restrictions. If the test statistics exceeds the test critical value, the null hypothesis is rejected: the restricted model is rejected in favor of the unrestrected model.

Choosing among the likelihood ratio test, Lagrange multiplier test, and Wald test is largely determined by computational cost:

- To conduct a likelihood ratio test, you need to estimate both the **restricted and unrestricted** models;
- To conduct a Score test (Lagrange multiplier test), you only need to estimate the **restricted** model (but the test requires an estimate of the variance–covariavnce matrix);
- To test a Wald test, you only need to estimate the **unrestricted** model (but the test requires an estimate of the variance–covariance matrix).