TA Session for Econometrics II 2025

6: Panel Data analysis

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November 20, 2025

1. Revision — Kronecker Product
2. Panel Data Models via the Kronecker Product
3. Panel Data Models

Kronecker Product: Definition

Kronecker Product:
行列 $A \in \mathbb{R}^{m\times n}$ と $B \in \mathbb{R}^{p\times q}$ に対して定義され，A **の各要素をスカラーとして** B **に掛けたブロック行列**として与えられる。

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

Dimention:

$$A \otimes B \in \mathbb{R}^{(mp)\times(nq)}.$$

Kronecker Product: Simple Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix}.$$

$$A \otimes B = \begin{bmatrix} 1B & 2B \\ 3B & 4B \end{bmatrix} = \begin{bmatrix} 0 & 5 & 0 & 10 \\ 6 & 7 & 12 & 14 \\ 0 & 15 & 0 & 20 \\ 18 & 21 & 24 & 28 \end{bmatrix}.$$

Kronecker Product: Useful Properties

- **Mixed-product property**

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

- **Transpose**

$$(A \otimes B)^\top = A^\top \otimes B^\top$$

- **Inverse**（可逆な場合）

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

- **Eigenvalues** A の固有値を λ_i，B の固有値を μ_j とすると

$$\lambda_i \mu_j \text{ が } A \otimes B \text{ の固有値}$$

Panel Data Structure and Kronecker Product Notation

Consider a balanced panel with N individuals observed over T periods. Let y_{it} denote the outcome and x_{it} the k-dimensional covariate vector.
Stacking observations yields

$$y = \begin{bmatrix} y_{11} \\ \vdots \\ y_{1T} \\ \vdots \\ y_{N1} \\ \vdots \\ y_{NT} \end{bmatrix} \in \mathbb{R}^{NT}.$$

Define the block-diagonal regressor matrix using the Kronecker product:

$$X = I_N \otimes X_i,$$

where

$$X_i = \begin{bmatrix} x_{i1}^\top \\ \vdots \\ x_{iT}^\top \end{bmatrix}.$$

This notation provides a compact way to express common panel estimators.

Pooled OLS Model

The pooled OLS model assumes a common intercept and constant slope parameters:

$$y_{it} = \alpha + x_{it}^{\top}\beta + u_{it}.$$

Stacking all observations yields

$$y = \iota_{NT}\alpha + X\beta + u.$$

The pooled OLS estimator is

$$\hat{\beta}_{\mathsf{POLS}} = (X^{\top}X)^{-1}X^{\top}(y - \iota_{NT}\hat{\alpha}),$$

with

$$\hat{\alpha} = \frac{1}{NT}\iota_{NT}^{\top}(y - X\hat{\beta}_{\mathsf{POLS}}).$$

This approach ignores all forms of unobserved individual effects.

Fixed Effects Model

The fixed effects model allows for individual-specific intercepts:

$$y_{it} = \alpha_i + x_{it}^\top \beta + u_{it}.$$

Let

$$D = I_N \otimes \iota_T,$$

so that

$$y = D\alpha + X\beta + u.$$

The within transformation removes α:

$$M = I_{NT} - (I_N \otimes T^{-1}\iota_T \iota_T^\top).$$

The estimator is then

$$\hat{\beta}_{\mathsf{FE}} = (X^\top M X)^{-1} X^\top M y.$$

The Kronecker product representation highlights the block-diagonal structure of the transformation.

Random Effects Model

The random effects model assumes

$$y_{it} = \alpha + x_{it}^\top \beta + c_i + u_{it},$$

where c_i is an individual-specific random component.
Let

$$c = (c_1 \iota_T, \ldots, c_N \iota_T)^\top = (I_N \otimes \iota_T)\, c^*,$$

where $c^* = (c_1, \ldots, c_N)^\top$.
The composite error is

$$v = c + u.$$

The GLS transformation involves the matrix

$$\Omega = \sigma_u^2 I_{NT} + \sigma_c^2 (I_N \otimes \iota_T \iota_T^\top).$$

The random effects estimator is

$$\hat{\beta}_{\mathsf{RE}} = (X^\top \Omega^{-1} X)^{-1} X^\top \Omega^{-1} y.$$

The Kronecker structure provides a compact expression for the covariance matrix and the RE transformation.

Summary of the Three Approaches

Pooled OLS:

$$y = \iota_{NT}\alpha + X\beta + u.$$

Fixed effects:

$$y = (I_N \otimes \iota_T)\alpha + X\beta + u, \qquad \hat{\beta}_{\mathsf{FE}} = (X^\top M X)^{-1} X^\top M y.$$

Random effects:

$$y = \iota_{NT}\alpha + X\beta + (I_N \otimes \iota_T)c^* + u, \qquad \hat{\beta}_{\mathsf{RE}} = (X^\top \Omega^{-1} X)^{-1} X^\top \Omega^{-1} y.$$

The Kronecker product allows all three estimators to be written using a unified matrix framework.

Panel Data Structure

Consider a balanced panel with N individuals and T time periods. For each individual $i = 1, \ldots, N$ and period $t = 1, \ldots, T$, let y_{it} denote the outcome and x_{it} the k-dimensional covariate vector.
The total number of observations is NT.
We start with the pooled OLS model, followed by fixed effects and random effects.

Pooled OLS Model

The pooled OLS model assumes

$$y_{it} = \alpha + x_{it}^\top \beta + u_{it}.$$

The estimator is obtained by minimising

$$\sum_{i=1}^{N} \sum_{t=1}^{T} (y_{it} - \alpha - x_{it}^\top \beta)^2.$$

The first-order conditions give

$$\hat{\beta}_{\text{POLS}} = \left(\sum_{i=1}^{N} \sum_{t=1}^{T} x_{it} x_{it}^\top \right)^{-1} \left(\sum_{i=1}^{N} \sum_{t=1}^{T} x_{it} (y_{it} - \hat{\alpha}) \right).$$

The pooled estimator ignores individual-specific unobserved effects.

Fixed Effects Model

The fixed effects model allows each individual to have its own intercept:

$$y_{it} = \alpha_i + x_{it}^\top \beta + u_{it}.$$

A common representation uses deviations from individual means. Define

$$\bar{y}_i = \frac{1}{T}\sum_{t=1}^{T} y_{it}, \qquad \bar{x}_i = \frac{1}{T}\sum_{t=1}^{T} x_{it}.$$

Subtracting individual means removes the individual intercept:

$$y_{it} - \bar{y}_i = (x_{it} - \bar{x}_i)^\top \beta + (u_{it} - \bar{u}_i).$$

The fixed effects estimator solves

$$\hat{\beta}_{\mathsf{FE}} = \left(\sum_{i=1}^{N}\sum_{t=1}^{T} (x_{it} - \bar{x}_i)(x_{it} - \bar{x}_i)^\top \right)^{-1} \left(\sum_{i=1}^{N}\sum_{t=1}^{T} (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i) \right).$$

Random Effects Model

The random effects model assumes

$$y_{it} = \alpha + x_{it}^\top \beta + c_i + u_{it},$$

where c_i is a random individual-specific effect.
The composite error is

$$v_{it} = c_i + u_{it}.$$

Let σ_c^2 and σ_u^2 denote the variances of c_i and u_{it}. Define the quasi-demeaning parameter

$$\theta = 1 - \left(\frac{\sigma_u^2}{\sigma_u^2 + T\sigma_c^2} \right)^{1/2}.$$

The transformed model is

$$y_{it} - \theta \bar{y}_i = (x_{it} - \theta \bar{x}_i)^\top \beta + (u_{it} - \theta \bar{u}_i).$$

The random effects estimator solves OLS on this transformed equation.

$$\hat{\beta}_{\mathsf{RE}} = \left(\sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \theta \bar{x}_i)(x_{it} - \theta \bar{x}_i)^\top \right)^{-1} \left(\sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \theta \bar{x}_i)(y_{it} - \theta \bar{y}_i) \right).$$

Summary

Pooled OLS:

$$y_{it} = \alpha + x_{it}^{\top}\beta + u_{it}.$$

Fixed effects:

$$y_{it} = \alpha_i + x_{it}^{\top}\beta + u_{it}.$$

Random effects:

$$y_{it} = \alpha + x_{it}^{\top}\beta + c_i + u_{it}.$$

The three estimators differ in how they handle unobserved heterogeneity across individuals.