Math Revision Session

Statistics (1): Probability

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1. Sets
2. Probability
   - Definition
   - Axioms
   - Addition Rule
   - Joint Probability
   - Conditional Probability
   - Independence
3. Bayes' Theorem
4. Random Variables

Sets and Their Definitions

- A **set** is a collection of distinct objects, called **elements**.
- A set is usually denoted by a capital letter, such as A, B, or C.
- Elements are written inside curly brackets.

Example:

$$A = \{1, 2, 3, 4, 5\}.$$

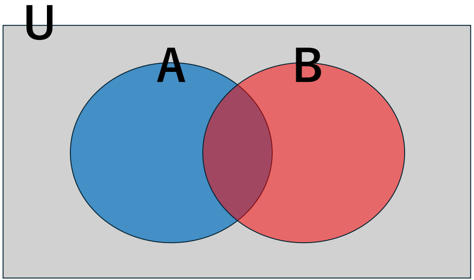
Types of Sets

- **Finite set**: a set with a limited number of elements.
- **Infinite set**: a set with infinitely many elements, such as $\mathbb{N}$.
- **Empty set** ($\emptyset$): a set with no elements.
- **Universal set** (U): the set containing all objects under consideration.

Set Operations

- **Union**: $A \cup B$ is the set of elements that belong to A or B (or both).
- **Intersection**: $A \cap B$ is the set of elements that belong to both A and B.
- **Difference**: $A - B$ is the set of elements in A but not in B.
- **Complement**: A^c is the set of elements not in A, relative to the universal set.

Example: Venn Diagram



- The overlapping region represents $A \cap B$.
- The entire coloured area represents $A \cup B$.

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Probability: Basic Ideas

Consider a roll of a fair die.

- **Sample space**:

  $$\Omega = \{1, 2, 3, 4, 5, 6\}$$

  This is the set of all possible outcomes.

- **Event**: a subset of the sample space. For example,

  $$A = \{1, 3, 5\}$$

  means "rolling an odd number."

- **Probability**: a function that assigns a number between 0 and 1 to each event.

Example: A Fair Die

For each $i \in \{1, 2, 3, 4, 5, 6\}$, define the event

$$A_i = \{i\}.$$

If the die is fair, then

$$\Pr(A_1) = \Pr(A_2) = \cdots = \Pr(A_6) = \frac{1}{6}.$$

Also,

$$\Pr(A_1) + \Pr(A_2) + \cdots + \Pr(A_6) = 1.$$

This reflects the idea that exactly one outcome occurs.

Probability Axioms

Kolmogorov's axioms:

- **Non-negativity**:

  $$\Pr(A) \geq 0$$

  for every event A.

- **Normalization**:

  $$\Pr(\Omega) = 1.$$

- **Additivity**: if A and B are mutually exclusive, then

  $$\Pr(A \cup B) = \Pr(A) + \Pr(B).$$

Consequences of the Axioms

- **Monotonicity**: if $A \subseteq B$, then

$$\Pr(A) \leq \Pr(B).$$

- **Complement rule**:

$$\Pr(A^c) = 1 - \Pr(A).$$

- **Finite additivity**: if $A_1, \ldots, A_n$ are mutually exclusive, then

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \Pr(A_i).$$

Addition Rule of Probability

For any two events A and B,

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B).$$

If A and B are mutually exclusive, then

$$\Pr(A \cap B) = 0,$$

so

$$\Pr(A \cup B) = \Pr(A) + \Pr(B).$$

Joint Probability

The **joint probability** of two events A and B is the probability that both events occur:

$$\Pr(A \cap B).$$

If A and B are independent, then

$$\Pr(A \cap B) = \Pr(A)\Pr(B).$$

Conditional Probability

The conditional probability of A given B is

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}, \qquad \Pr(B) > 0.$$

This tells us how the probability of A changes once we know that B has occurred.

Independence of Events

Two events A and B are **independent** if

$$\Pr(A \cap B) = \Pr(A)\Pr(B).$$

This means that knowing one event occurs does not change the probability of the other.

If A and B are independent, then

$$\Pr(A \mid B) = \Pr(A).$$

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Bayes' Theorem

Bayes' theorem states that

$$\Pr(A \mid B) = \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B)}, \qquad \Pr(B) > 0.$$

It expresses the probability of A given B in terms of:

- the prior probability $\Pr(A)$,
- the conditional probability $\Pr(B \mid A)$,
- and the overall probability $\Pr(B)$.

Derivation of Bayes' Theorem

From the definition of conditional probability,

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}, \qquad \Pr(B \mid A) = \frac{\Pr(A \cap B)}{\Pr(A)}.$$

Hence,

$$\Pr(A \cap B) = \Pr(A \mid B)\Pr(B) = \Pr(B \mid A)\Pr(A).$$

Therefore,

$$\Pr(A \mid B) = \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B)}.$$

Law of Total Probability

If A and A^c form a partition, then

$$\Pr(B) = \Pr(B \mid A)\Pr(A) + \Pr(B \mid A^c)\Pr(A^c).$$

Substituting this into Bayes' theorem gives

$$\Pr(A \mid B) = \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B \mid A)\Pr(A) + \Pr(B \mid A^c)\Pr(A^c)}.$$

Example

In a hospital:

- $\Pr(A) = 0.01$: a person has the disease.
- $\Pr(B \mid A) = 0.99$: the test is positive given the disease.
- $\Pr(B \mid A^c) = 0.05$: the test is positive without the disease.

Question: What is

$$\Pr(A \mid B)?$$

That is, what is the probability that a person who tested positive actually has the disease?

Computing $\Pr(B)$

Using the law of total probability,

$$\Pr(B) = \Pr(B \mid A)\Pr(A) + \Pr(B \mid A^c)\Pr(A^c).$$

So,

$$\Pr(B) = (0.99)(0.01) + (0.05)(0.99) = 0.0099 + 0.0495 = 0.0594.$$

Applying Bayes' Theorem

Now

$$\Pr(A \mid B) = \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B)} = \frac{(0.99)(0.01)}{0.0594}.$$

Hence,

$$\Pr(A \mid B) \approx 0.1667.$$

Therefore, the probability that a person who tested positive actually has the disease is about

$$16.67\%.$$

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What is a Random Variable?

A **random variable** is a function that assigns a real number to each outcome in the sample space of a random experiment.

- Random variables are usually denoted by capital letters such as X, Y, and Z.
- Their values depend on the outcome of the experiment.
- A probability distribution describes how probabilities are assigned to their possible values.

Types of Random Variables

Random variables are usually divided into two types:

- **Discrete random variable**: takes a finite or countable number of values.
- **Continuous random variable**: takes infinitely many values over an interval.

Example of a Discrete Random Variable

Suppose X is the outcome of rolling a fair die. Then

$$X \in \{1, 2, 3, 4, 5, 6\}.$$

Since the die is fair,

$$\Pr(X = x) = \frac{1}{6}, \qquad x \in \{1, 2, 3, 4, 5, 6\}.$$

So X is a discrete random variable.

Example of a Continuous Random Variable

Suppose Y is the height of a randomly selected person.
Then Y can take infinitely many values in some interval, so it is a continuous random variable.

Its distribution is described by a probability density function (PDF), which we may write informally as

$$f_Y(y).$$

The probability of Y taking values in an interval is obtained by integrating the density over that interval.

Summary

In this lecture, we studied:

- sets and set operations,
- sample spaces and events,
- probability axioms,
- addition rule, joint probability, conditional probability, and independence,
- Bayes' theorem,
- discrete and continuous random variables.

These ideas are the foundation of probability theory and statistics.