# Math Revision Session

## Statistics (3): Continuous Random Variables and Famous Distributions

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April 8, 2026

1. Continuous Random Variables
2. CDF and PDF
3. Uniform Distribution
4. Expectation and Variance
5. Joint and Marginal Distributions
6. Covariance and Correlation
7. Independence
8. Conditional Density and Conditional Expectation
9. Summary

# Continuous Random Variable

A **continuous random variable** can take infinitely many possible values within an interval.

Examples:

- height,
- weight,
- temperature,
- time.

Unlike discrete random variables, continuous random variables are not counted one by one.

# Probability in the Continuous Case

For a continuous random variable, probability is assigned to **intervals**, not to single points.

**Discrete case:**

$$\Pr(X = 3) = \frac{1}{6} \qquad \text{for a fair die.}$$

**Continuous case:**

$$\Pr(X = 1.5) = 0.$$

Instead, we calculate probabilities such as

$$\Pr(a \leq X \leq b) = \int_a^b f_X(x)\, dx.$$

# Why Does $\Pr(X = x) = 0$?

In the continuous case, a single point has no width, so its probability is zero:

$$\Pr(X = x) = 0.$$

Therefore,

$$\Pr(0 \le X \le 1) = \Pr(0 < X < 1) = \Pr(0 \le X < 1) = \Pr(0 < X \le 1).$$

Including or excluding endpoints does not change the probability.

# Cumulative Distribution Function (CDF)

The **cumulative distribution function** (CDF) of $X$ is

$$F_X(x) = \Pr(X \leq x).$$

Properties:

- $F_X(x)$ is non-decreasing,
- $\lim_{x \to -\infty} F_X(x) = 0$,
- $\lim_{x \to \infty} F_X(x) = 1$.

# Probability Density Function (PDF)

If $X$ is continuous and $F_X$ is differentiable, then the **probability density function** (PDF) is

$$f_X(x) = \frac{d}{dx} F_X(x).$$

The PDF is not itself a probability. Instead,

$$\Pr(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(x)\,dx.$$

# Properties of the PDF

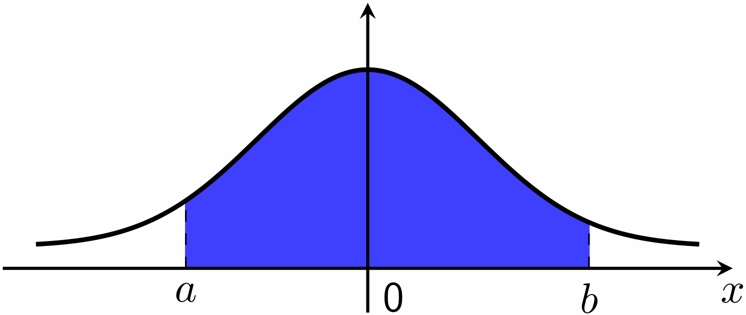
A probability density function must satisfy:

- $$f_X(x) \geq 0 \qquad \text{for all } x,$$
- $$\int_{-\infty}^{\infty} f_X(x)\, dx = 1.$$

Also,

$$\Pr(X = x) = 0 \qquad \text{for every single point } x.$$

# Graphical Interpretation

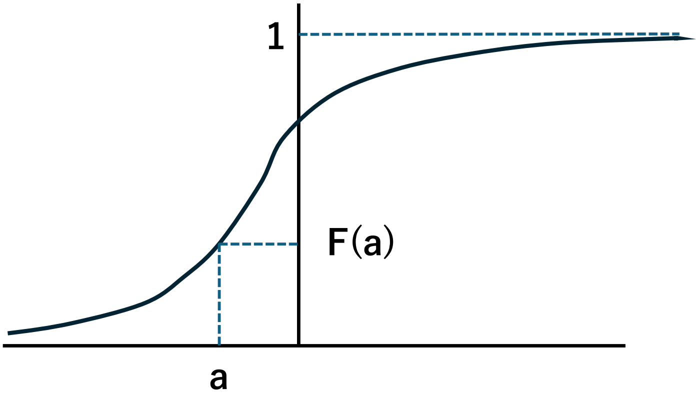



a PDF

The shaded area represents

$$\Pr(a \leq X \leq b).$$

# A CDF



# Uniform Distribution

A random variable $X$ follows a **continuous uniform distribution** on $[a, b]$ if every point in the interval is equally likely in terms of density.

Its PDF is

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b, \\ 0, & \text{otherwise.} \end{cases}$$

Its CDF is

$$F_X(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \leq x \leq b, \\ 1, & x > b. \end{cases}$$

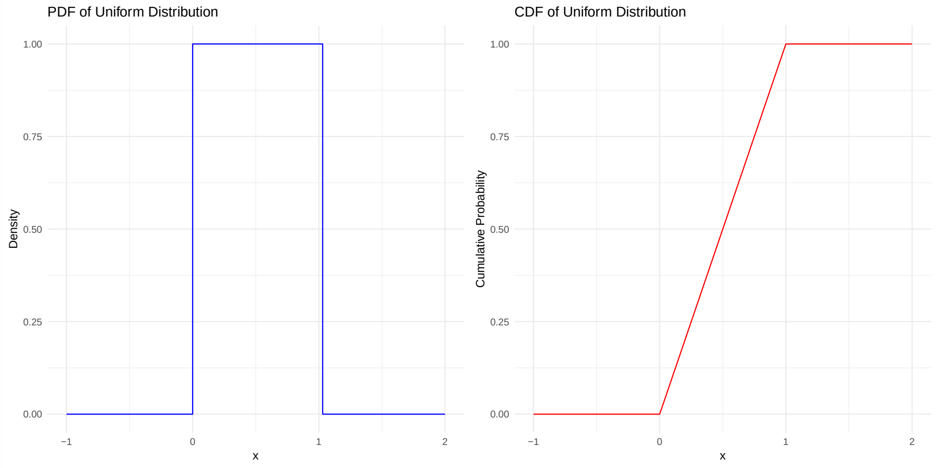
# Example: Uniform$(0, 1)$

If $X \sim \mathrm{Uniform}(0, 1)$, then

$$f_X(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

This means that probabilities are proportional to interval length.

# Graph of the Uniform Distribution




The PDF is constant on the interval $[a, b]$.

# Expectation and Variance of Continuous Random Variables

If $X$ has PDF $f_X(x)$, then its expectation is

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

Its variance is

$$\mathrm{Var}(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2,$$

where

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx.$$

# Example: Uniform Distribution

If $X \sim \text{Uniform}(a, b)$, then

$$E[X] = \frac{a+b}{2}, \qquad \text{Var}(X) = \frac{(b-a)^2}{12}.$$

So the mean is the midpoint of the interval, and the variance depends on the width of the interval.

# A Simple Observation

The variance measures how spread out the values of $X$ are around the mean.
If

$$\mathrm{Var}(X) = 0,$$

then $X$ is a constant with probability 1.

# Joint Distribution of Continuous Random Variables

For two continuous random variables $X$ and $Y$, the **joint cumulative distribution function** is

$$F_{X,Y}(x,y) = \Pr(X \leq x,\ Y \leq y).$$

If this function is differentiable, then the **joint density** is

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x\, \partial y} F_{X,Y}(x,y).$$

# Probability from the Joint Density

For a rectangle $[a, b] \times [c, d]$,

$$\Pr(a \leq X \leq b,\ c \leq Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x, y)\, dy\, dx.$$

So the joint density describes how probability is distributed over a two-dimensional region.

# Marginal Distributions

The marginal density of $X$ is obtained by integrating out $Y$:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy.$$

Similarly, the marginal density of $Y$ is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx.$$

# Covariance

The covariance between $X$ and $Y$ is

$$\mathrm{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])].$$

For continuous random variables,

$$\mathrm{Cov}(X, Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - E[X])(y - E[Y]) f_{X,Y}(x, y)\, dx\, dy.$$

Covariance measures the extent to which two variables move together.

# Correlation Coefficient

The correlation coefficient is

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y},$$

where

$$\sigma_X = \sqrt{\mathrm{Var}(X)}, \qquad \sigma_Y = \sqrt{\mathrm{Var}(Y)}.$$

It measures the strength and direction of the linear relationship between $X$ and $Y$.

# Independence of Continuous Random Variables

Two continuous random variables $X$ and $Y$ are independent if

$$f_{X,Y}(x, y) = f_X(x) f_Y(y) \qquad \text{for all } x, y.$$

Equivalently,

$$F_{X,Y}(x, y) = F_X(x) F_Y(y) \qquad \text{for all } x, y.$$

# Independence and Correlation

If $X$ and $Y$ are independent, then

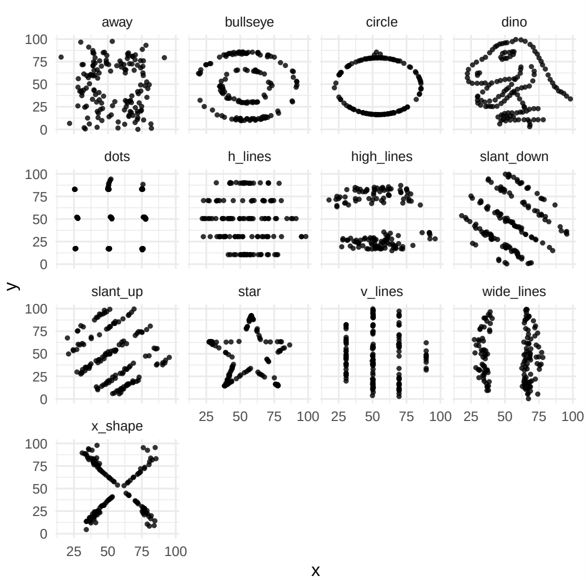
$$\mathrm{Cov}(X, Y) = 0 \qquad \text{and hence} \qquad \rho(X, Y) = 0.$$

However,

$$\rho(X, Y) = 0 \quad \not\Rightarrow \quad X \text{ and } Y \text{ are independent.}$$

Zero correlation means no linear relationship, but nonlinear dependence may still exist.

# The Datasaurus Dozen



# Conditional Density

For continuous random variables $X$ and $Y$, the conditional density of $X$ given $Y = y$ is

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}, \qquad \text{provided } f_Y(y) > 0.$$

This is the continuous analogue of conditional probability in the discrete case.

# Conditional Expectation

The conditional expectation of $X$ given $Y = y$ is

$$E[X \mid Y = y] = \int_{-\infty}^{\infty} x \, f_{X|Y}(x \mid y) \, dx.$$

It gives the average value of $X$ when we know that $Y = y$.

# Law of Total Expectation

A fundamental rule is the law of total expectation:

$$E[X] = E[E[X \mid Y]].$$

This says that the unconditional expectation can be obtained by averaging the conditional expectation.

# Summary

In this lecture, we studied:

- continuous random variables,
- CDF and PDF,
- uniform distribution,
- expectation and variance,
- joint and marginal distributions,
- covariance and correlation,
- independence,
- conditional density and conditional expectation.