Math Revision Session

1: Matrix Algebra (1)

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April 6, 2026

Course Information

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- Room: Refer to CLE.
- Day and Time of Classes: Refer to CLE.
- Number of Classes: 5 or 6 sessions (90 minutes each)

All handouts will be uploaded on my website[1] after the final lesson.

[1] Website: `https://jukinahatakeyama.github.io/jhatakeyama/`

Introduction

Matrices are useful because they can efficiently represent and manipulate large amounts of data, simplify calculations, and organise complex equations in a compact form.

For example, in undergraduate econometrics, we consider the following linear model:

$$
\begin{aligned}
y_i &= \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \cdots + \beta_K x_{i,K} + u_i && (1) \\
&= \beta_0 + \sum_{k=1}^{K} \beta_k x_{i,k} + u_i && (2)
\end{aligned}
$$

for $i = 1, \ldots, N$.

Stacked Representation

When we stack these equations for $i = 1, \ldots, N$, we obtain

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} 1 & x_{1,1} & x_{1,2} & \ldots & x_{1,K} \\ 1 & x_{2,1} & x_{2,2} & \ldots & x_{2,K} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{N,1} & x_{N,2} & \ldots & x_{N,K} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_K \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}.$$

This can be written more compactly as

$$y = X\beta + u,$$

where

$$y, u \in \mathbb{R}^N, \qquad X \in \mathbb{R}^{N \times (K+1)}, \qquad \beta \in \mathbb{R}^{K+1}.$$

$\mathbb{R}$ denotes the set of real numbers.

Dimension

If x is ...

- a scalar, $x \in \mathbb{R}$.
  This means that x is a real number.
  Example: $x = 2 \in \mathbb{R}$
- a vector, $x \in \mathbb{R}^m$, where m is a positive integer.
  This means that x has m elements.
  Example:

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \mathbb{R}^3$$

- a matrix, $x \in \mathbb{R}^{\ell \times m}$, where ℓ and m are positive integers.
  This means that x has $\ell \times m$ entries.
  Example:
$$x = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \in \mathbb{R}^{3\times 2}$$

Here, ℓ is the number of rows and m is the number of columns.

General Representation of a Matrix

Let X be a matrix of dimension $a \times b$. The element of X in the i-th row and j-th column is denoted by x_{ij}, where $i = 1, \ldots, a$ and $j = 1, \ldots, b$. Then we can write

$$X = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1b} \\ x_{21} & x_{22} & \dots & x_{2b} \\ \vdots & \vdots & \ddots & \vdots \\ x_{a1} & x_{a2} & \dots & x_{ab} \end{pmatrix} \in \mathbb{R}^{a \times b}.$$

Each element x_{ij} represents the value in the i-th row and j-th column.

A vector can be viewed as an $a \times 1$ matrix.

Square Matrix

A square matrix is a matrix with the same number of rows and columns. In other words, a matrix of size $n \times n$ is called a square matrix.

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

This is a 3×3 square matrix.

Symmetric Matrix

A matrix A is called symmetric if it is equal to its own transpose:

$$A^\top = A.$$

For a matrix $A = [a_{ij}]$, this is equivalent to

$$a_{ij} = a_{ji} \qquad \text{for all } i, j.$$

So the entries are mirrored along the main diagonal.

Example of a Symmetric Matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

This matrix is symmetric because

$$A^\top = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} = A.$$

Diagonal Matrix

A matrix $D \in \mathbb{R}^{n \times n}$ is called a diagonal matrix if all off-diagonal entries are zero:

$$D = \begin{pmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ 0 & 0 & d_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_n \end{pmatrix}.$$

Example:

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

Diagonal matrices are useful for matrix powers, eigenvalue problems, and diagonalisation.

Identity Matrix

The identity matrix is a square matrix with 1s on the main diagonal and 0s elsewhere. It is denoted by I_n.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The identity matrix satisfies

$$I_n A = A I_n = A$$

whenever the multiplication is defined.

Idempotent Matrix

A matrix A is called **idempotent** if

$$A^2 = A.$$

Example:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Then

$$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = A.$$

Symmetric and Idempotent Matrices

- A is **symmetric** if
$$A^\top = A.$$
- A is **idempotent** if
$$A^2 = A.$$
- The identity matrix is both symmetric and idempotent:
$$I^\top = I, \qquad I^2 = I.$$

Matrix Addition

If A and B are matrices of the same dimension $m \times n$, then their sum $C = A + B$ is defined entry by entry:

$$C_{ij} = A_{ij} + B_{ij}.$$

Example of Matrix Addition

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

Then

$$A + B = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}.$$

Matrix Subtraction

If A and B are matrices of the same dimension, then their difference $C = A - B$ is defined by

$$C_{ij} = A_{ij} - B_{ij}.$$

Example of Matrix Subtraction

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

Then

$$A - B = \begin{pmatrix} 1-5 & 2-6 \\ 3-7 & 4-8 \end{pmatrix} = \begin{pmatrix} -4 & -4 \\ -4 & -4 \end{pmatrix}.$$

Properties of Matrix Addition

Let $A, B, C \in \mathbb{R}^{m \times n}$. Then:

1. $A + B = B + A$ (commutative)
2. $(A + B) + C = A + (B + C)$ (associative)
3. $A + \mathbf{0} = A$ (additive identity)
4. $A - A = \mathbf{0}$

Zero Matrix

The zero matrix is a matrix whose entries are all zero:

$$\mathbf{0} = \begin{pmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}.$$

Scalar Multiplication

Let A be an $m \times n$ matrix and c a scalar. Then cA is obtained by multiplying each entry of A by c:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad \Rightarrow \quad cA = \begin{pmatrix} ca_{11} & ca_{12} & \dots & ca_{1n} \\ ca_{21} & ca_{22} & \dots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \dots & ca_{mn} \end{pmatrix}.$$

Properties of Scalar Multiplication

For matrices A, B of the same size and scalars c, d:

1. $c(A + B) = cA + cB$
2. $(c + d)A = cA + dA$
3. $c(dA) = (cd)A$
4. $1A = A$

Exercise: Scalar Multiplication

Solve the following:

1. Compute
$$3\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$
2. Verify
$$2\left(\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}+\begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix}\right)=2\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}+2\begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix}.$$
3. Show that $c(dA)=(cd)A$ for $c=2$, $d=3$, and
$$A=\begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}.$$

Matrix Operations Practice

Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \qquad B = \begin{pmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix}, \qquad c = 3.$$

Compute:

1. $A + B$
2. $B - A$
3. $3A$
4. $2A + cB$

Matrix Multiplication

For two matrices A and B, the product AB is defined only when the number of columns of A equals the number of rows of B.

If

$$A \in \mathbb{R}^{m \times n}, \qquad B \in \mathbb{R}^{n \times p},$$

then

$$AB \in \mathbb{R}^{m \times p}.$$

In matrix calculations, it is always important to check the dimensions first. The dimensions must be compatible.

Formula for Matrix Multiplication

If $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$, then

$$AB = \begin{pmatrix} \sum_{k=1}^{n} a_{1k}b_{k1} & \sum_{k=1}^{n} a_{1k}b_{k2} & \cdots & \sum_{k=1}^{n} a_{1k}b_{kp} \\ \sum_{k=1}^{n} a_{2k}b_{k1} & \sum_{k=1}^{n} a_{2k}b_{k2} & \cdots & \sum_{k=1}^{n} a_{2k}b_{kp} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} a_{mk}b_{k1} & \sum_{k=1}^{n} a_{mk}b_{k2} & \cdots & \sum_{k=1}^{n} a_{mk}b_{kp} \end{pmatrix}.$$

Example of Matrix Multiplication

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

Then

$$AB = \begin{pmatrix} 1 \cdot 5 + 2 \cdot 7 & 1 \cdot 6 + 2 \cdot 8 \\ 3 \cdot 5 + 4 \cdot 7 & 3 \cdot 6 + 4 \cdot 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}.$$

Properties of Matrix Multiplication

Let A, B, C be matrices of compatible dimensions and $\alpha \in \mathbb{R}$. Then:

1. $\alpha(AB) = (\alpha A)B$
2. $(AB)C = A(BC)$
3. $A(B + C) = AB + AC$
4. $(A + B)C = AC + BC$
5. In general, $AB \neq BA$

Identity Matrix Multiplication

For the identity matrix I_n,

$$I_n A = A I_n = A$$

whenever the multiplication is defined.

Example:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$$

Then

$$I_2 A = A.$$

Transpose of a Matrix

The transpose of a matrix A, denoted by $A^\top$, is obtained by interchanging rows and columns.
If A is an $m \times n$ matrix, then $A^\top$ is an $n \times m$ matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad \Rightarrow \quad A^\top = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}.$$

Example of a Transpose

Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

Then

$$A^\top = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

In some contexts, A' is also used to denote the transpose.

Matrix Representation of the Model

Consider again the linear model

$$y_i = \beta_0 + \sum_{k=1}^{K} \beta_k x_{i,k} + u_i.$$

Let

$$x_i = \begin{pmatrix} x_{i,1} \\ x_{i,2} \\ \vdots \\ x_{i,K} \end{pmatrix} \in \mathbb{R}^K, \qquad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_K \end{pmatrix} \in \mathbb{R}^K.$$

Then the model can be written as

$$y_i = \beta_0 + x_i^\top \beta + u_i,$$

where $x_i^\top \beta = \sum_{k=1}^{K} \beta_k x_{i,k}$.

Properties of the Transpose

For compatible matrices A, B, scalar α, and vector x:

1. $(A^\top)^\top = A$
2. $(\alpha A)^\top = \alpha A^\top$
3. $(A + B)^\top = A^\top + B^\top$
4. $(AB)^\top = B^\top A^\top$
5. $x^\top x = \sum_{i=1}^{n} x_i^2$

A Useful Example

If

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

where each a_i is a $1 \times k$ row vector, then

$$A^\top = (a_1^\top, a_2^\top, \ldots, a_n^\top).$$

So the rows of A become the columns of $A^\top$.

Why is $X^\top X$ Symmetric?

Let X be an $n \times k$ matrix. Then $X^\top$ is a $k \times n$ matrix.
We want to show that $X^\top X$ is symmetric:

$$(X^\top X)^\top = X^\top X.$$

Using $(AB)^\top = B^\top A^\top$,

$$(X^\top X)^\top = X^\top (X^\top)^\top = X^\top X.$$

Therefore, $X^\top X$ is symmetric.

Determinant of a Matrix

The determinant is a scalar value computed from a square matrix.

- For
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \det(A) = ad - bc.$$
- For
$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix},$$
$$\det(A) = a(ei - fh) - b(di - fg) + c(dh - eg).$$

Properties of the Determinant

For a square matrix A:

1. If $\det(A) = 0$, then A is singular and does not have an inverse.
2. If $\det(A) \neq 0$, then A is non-singular and has an inverse.
3. $\det(I) = 1$.
4. $\det(AB) = \det(A)\det(B)$.

Inverse Matrix

The inverse of a matrix A, denoted by A^{-1}, is the matrix satisfying

$$AA^{-1} = A^{-1}A = I.$$

A square matrix has an inverse if and only if

$$\det(A) \neq 0.$$

What Does the Existence of A^{-1} Mean?

If A^{-1} exists, then the linear transformation by A can be undone:

$$A^{-1}A = AA^{-1} = I.$$

So A^{-1} is not just "division by a matrix". It is the matrix that reverses the transformation represented by A.

If A^{-1} does not exist, then some information is lost and the original vector cannot always be recovered uniquely.

Formula for the Inverse

If A is a square matrix with $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A),$$

where $\operatorname{adj}(A)$ is the adjugate matrix.
For a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we have

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \qquad \text{if } ad - bc \neq 0.$$

Example of an Inverse

Let

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}.$$

Then

$$\det(A) = 2 \cdot 4 - 1 \cdot 3 = 5.$$

Hence,

$$A^{-1} = \frac{1}{5} \begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}.$$

Matrix Types: Definite and Semi-Definite

For a symmetric matrix $A \in \mathbb{R}^{n \times n}$, the sign of the quadratic form

$$x^\top A x$$

is important.

- **Positive definite:**
$$x^\top A x > 0 \qquad \text{for all } x \neq 0$$
- **Positive semidefinite:**
$$x^\top A x \geq 0 \qquad \text{for all } x$$
- **Negative definite:**
$$x^\top A x < 0 \qquad \text{for all } x \neq 0$$
- **Negative semidefinite:**
$$x^\top A x \leq 0 \qquad \text{for all } x$$
- **Indefinite:** $x^\top A x$ can take both positive and negative values.

Positive Definite Matrix

A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is called **positive definite** if

$$x^\top Ax > 0 \qquad \text{for all } x \neq 0.$$

If

$$x^\top Ax \geq 0 \qquad \text{for all } x,$$

then A is called **positive semidefinite**.

How Can We Check It?

There are several ways to check whether a symmetric matrix is positive definite.

- **Directly from the quadratic form:** compute $x^\top A x$ and check whether it is always positive for $x \neq 0$.
- **Eigenvalues:** A is positive definite if and only if all eigenvalues of A are positive.
- **Leading principal minors:** for a symmetric matrix, A is positive definite if all leading principal minors are positive.

For 2×2 matrices, these checks are often simple.

Example: A 2×2 Matrix

Consider

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

For $x = (x_1, x_2)^\top$,

$$x^\top A x = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2x_1^2 + 2x_1x_2 + 2x_2^2.$$

We can rewrite this as

$$x_1^2 + (x_1 + x_2)^2 + x_2^2 > 0 \qquad \text{for all } x \neq 0.$$

Therefore, A is positive definite.

A Useful Criterion for 2×2 Matrices

For a symmetric matrix

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

A is positive definite if

$$a > 0 \qquad \text{and} \qquad \det(A) = ac - b^2 > 0.$$

This is a special case of the leading principal minor condition.

Why is a Positive Definite Matrix Important?

- A positive definite matrix is always **invertible**.
- Since $x^\top Ax > 0$ for all $x \neq 0$, it plays an important role in **quadratic optimisation**.
- In econometrics,
$$\hat{\beta} = (X^\top X)^{-1} X^\top y$$
requires $X^\top X$ to be invertible.
- In statistics, covariance matrices satisfy
$$a^\top \Sigma a = \mathrm{Var}(a^\top X) \geq 0.$$

Therefore, positive definite and positive semidefinite matrices play an important role in linear algebra, statistics, and econometrics.