Math Revision Session

2: Matrix Algebra (2)

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1. Rank
2. Eigenvalues and Eigenvectors
3. Diagonalisation

Matrix Rank

Definition:
The rank of a matrix A is the maximum number of **linearly independent** rows or columns in A.

Key Properties:

- $\text{rank}(A) \leq \min(m, n)$ for $A \in \mathbb{R}^{m \times n}$.
- $\text{rank}(A) = \text{rank}(A^\top)$.
- A is **full rank** if $\text{rank}(A) = \min(m, n)$.
- If $\text{rank}(A) < \min(m, n)$, then some rows or columns are linearly dependent.

Linear Independence and Rank

Let

$$A = \begin{pmatrix} \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \end{pmatrix} \in \mathbb{R}^{m \times n},$$

where $a_1, \ldots, a_n$ are the columns of A.

The columns of A are linearly independent if

$$c_1 a_1 + c_2 a_2 + \cdots + c_n a_n = 0$$

implies

$$c_1 = c_2 = \cdots = c_n = 0.$$

In matrix form, this is

$$Ac = 0 \quad \Rightarrow \quad c = 0.$$

Therefore,

Examples of Linear Independence

Example 1: Linearly Independent

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

If

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

then $c_1 = 0$ and $c_2 = 0$. Hence, they are linearly independent.

Example 2: Linearly Dependent

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Since $\mathbf{v}_2 = 2\mathbf{v}_1$, they are linearly dependent.

Examples of Matrix Rank

Example 1:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Using row operations,

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

So, $\mathrm{rank}(A) = 2$.

Example 2:

$$B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 3 & 0 & 6 \end{pmatrix}$$

Since row 2 is zero and row 3 is a multiple of row 1, $\mathrm{rank}(B) = 1$.

Why is Rank Important?

- Rank tells us how many rows or columns contain **independent information**.
- A square matrix is invertible if and only if it is **full rank**.
- If a matrix is **rank deficient**, some rows or columns are redundant.
- In regression analysis, if the columns of X are linearly dependent, then $X^\top X$ is singular.
- Rank is also related to **degrees of freedom**. In many linear models,

$$\text{residual degrees of freedom} = n - \mathrm{rank}(X).$$

Eigenvalues and Eigenvectors

Given a square matrix $A \in \mathbb{R}^{n\times n}$, a scalar λ and a nonzero vector $x \in \mathbb{R}^n$ satisfy

$$Ax = \lambda x.$$

- λ is called an **eigenvalue** of A.
- x is called an **eigenvector** corresponding to λ.

This means that multiplying x by A changes only its **length**, not its **direction**.

Intuition

- Usually, multiplying a vector by a matrix changes both its direction and its length.
- For an eigenvector, the direction stays the same.
- The eigenvalue tells us how much the vector is scaled.

$$Ax = \lambda x$$

- If $\lambda > 1$, the vector is stretched.
- If $0 < \lambda < 1$, the vector is shrunk.
- If $\lambda < 0$, the direction is reversed.

How to Find Eigenvalues and Eigenvectors

Step 1: Find eigenvalues
Solve

$$\det(A - \lambda I) = 0.$$

Step 2: Find eigenvectors
For each eigenvalue λ, solve

$$(A - \lambda I)x = 0.$$

If A is symmetric, then:

- all eigenvalues are real;
- eigenvectors corresponding to different eigenvalues are orthogonal.

Example: Finding Eigenvalues

Consider

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}.$$

We solve

$$\det(A - \lambda I) = \det \begin{pmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix} = 0.$$

So,

$$(4-\lambda)(3-\lambda) - 2 = 0,$$

$$\lambda^2 - 7\lambda + 10 = 0.$$

Hence, the eigenvalues are

$$\lambda = 5, \qquad \lambda = 2.$$

Example: Finding Eigenvectors

For $\lambda = 5$,

$$(A - 5I)x = 0 \quad \Rightarrow \quad \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

So, one eigenvector is

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $\lambda = 2$,

$$(A - 2I)x = 0 \quad \Rightarrow \quad \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

So, one eigenvector is

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Diagonalisation

A square matrix A is called **diagonalisable** if there exist an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

Here,

- the columns of P are eigenvectors of A;
- the diagonal entries of D are the corresponding eigenvalues.

When is Diagonalisation Possible?

A matrix $A \in \mathbb{R}^{n \times n}$ is diagonalisable if it has n linearly independent eigenvectors.

In particular:

- if A has n distinct eigenvalues, then it is diagonalisable;
- if A is symmetric, then it is always diagonalisable.

Why is Diagonalisation Useful?

If

$$A = PDP^{-1},$$

then

$$A^k = PD^kP^{-1}.$$

This is useful because powers of a diagonal matrix are easy to compute:

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \Rightarrow \quad D^k = \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix}.$$

So diagonalisation makes repeated multiplication much simpler.

Example of Diagonalisation

Consider

$$A = \begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix}.$$

The characteristic equation is

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 1 \\ 6 & 3-\lambda \end{vmatrix} = 0.$$

So,

$$(4-\lambda)(3-\lambda) - 6 = 0$$
$$\lambda^2 - 7\lambda + 6 = 0,$$

and hence

$$\lambda_1 = 6, \qquad \lambda_2 = 1.$$

Constructing P and D

For $\lambda_1 = 6$,

$$(A - 6I)x = 0 \quad \Rightarrow \quad \begin{pmatrix} -2 & 1 \\ 6 & -3 \end{pmatrix} x = 0,$$

so one eigenvector is

$$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

For $\lambda_2 = 1$,

$$(A - I)x = 0 \quad \Rightarrow \quad \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix} x = 0,$$

so one eigenvector is

$$v_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Therefore,

$$P = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}, \qquad D = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using Diagonalisation to Compute Powers

Since

$$A = PDP^{-1},$$

we have

$$A^5 = PD^5P^{-1}.$$

Because

$$D = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix},$$

it follows that

$$D^5 = \begin{pmatrix} 6^5 & 0 \\ 0 & 1^5 \end{pmatrix} = \begin{pmatrix} 7776 & 0 \\ 0 & 1 \end{pmatrix}.$$

So, computing A^5 is much easier through D^5 than by multiplying A by itself five times.